Endogenous Ranking and Equilibrium Lorenz Curve Across (ex-ante) Identical Countries

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Abstract: This paper proposes a symmetry-breaking model of trade with a (large but) finite number of (ex-ante) identical countries and a continuum of tradeable goods, which differ in their dependence on local differentiated producer services. Productivity differences across countries arise endogenously through free entry to the local service sector in each country. In any stable equilibrium, the countries sort themselves into specializing in different sets of tradeable goods and a strict ranking of countries in per capita income, TFP, and the capital-labor ratio emerge endogenously. Furthermore, the distribution of country shares, the Lorenz curve, becomes analytically solvable in the limit, as the number of countries grows unbounded. Using this limit as an approximation allows us to study what determines the shape of distribution, perform various comparative statics and to evaluate the welfare effects of trade.

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1.   Introduction

Rich countries tend to have higher TFPs and higher capital-labor ratios than the poor. Such empirical regularities are generally viewed as a causality running from TFPs and/or capital-labor ratios to per capita income. However, there is a complementary approach, popular in trade and economic geography, that suggests a two-way causality. According to this approach, trade (and factor mobility) among countries/regions, even if they were ex-ante identical, could lead to the instability of the symmetric equilibrium in which they would remain identical. With such *symmetry-breaking,* cross-sectional dispersion and correlation in per capita income, TFPs, and capital-labor ratios, emerge endogenously as only stable patterns.[2] This approach does not try to argue that countries/regions are ex-ante identical nor that any exogenous heterogeneity or country/region-specific shocks are unimportant. On the contrary, it suggests that even small heterogeneity or shocks may be important, as they could be amplified to create large productivity and income differences, which makes this approach appealing to many as an explanation for "Great Divergence" and "Growth Miracles."

The existing studies of symmetry-breaking, however, demonstrate this insight in a two-country/region setup, which makes it unclear what the message of this approach is when applied to a multi-country/region world. For example, does a symmetry-breaking mechanism split the world into the rich and poor clusters, as the narrative of this literature, such as "core-periphery" or "polarization" might suggest? Or does it keep splitting the world into finer clusters until the distribution becomes more disperse, possibly generating a power-law like distribution, as observed in the size distribution of metropolitan areas? More generally, which underlying features of the world determine the shape of the distribution? Not only the existing studies on symmetry-breaking are unable to answer these questions, but also generate little analytical results on comparative statics and welfare effects. The aim of this paper is to propose an analytically tractable symmetry-breaking model of trade as a framework in which one would address these issues.

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[2] See Fujita, Krugman, and Venables (1999) and Combes, Mayer, and Thisse (2008) in economic geography and Ethier (1982b), Helpman (1986, p.344-346), Krugman and Venables (1995) and Matsuyama (1996) in international trade. The view that trade itself could magnify inequality among nations was discussed informally by Myrdal (1957) and Lewis (1977). See Matsuyama (2011) for more references. *Symmetry-breaking* is a circular mechanism that generates stable asymmetric outcomes in the symmetric environment due to the instability of the symmetric outcome. Although most prominent in economic geography, it has found applications in other areas of economics: see a New Palgrave entry on "symmetry-breaking" by Matsuyama (2008) as well as a related entry on "emergence" by Ioannides (2008).

More specifically, imagine a world with a (large but) finite number of (ex-ante) identical countries. In each country, the representative household supplies a single composite of primary factors and has Cobb-Douglas preferences over a continuum of tradeable goods, as in Dornbusch, Fischer, and Samuelson (1977). Productivity in each country is endogenous and depends on the available variety of local differentiated producer services, determined by free entry to the local service sector, as in Dixit and Stiglitz (1977) model of monopolistic competition. One key assumption is that tradeable sectors differ in their dependence on local services. This creates a circular mechanism between patterns of trade and cross-country productivity differences. Having more variety of local services not only makes a country more productive. It also gives a country comparative advantage in tradeable sectors that are more dependent on those services. This in turn means a larger market for services, hence more firms enter to provide such services. As a result, the country ends up having more variety of local services and become more productive.

With (a continuum of) tradable goods vastly outnumbering (a finite number of) countries, this circular mechanism sorts different countries into specializing in different sets of tradeable goods (*endogenous comparative advantage*) and leads to a *strict ranking* of countries in income, TFP, and (in an extension that allows for variable factor supply) capital-labor ratio in any stable equilibrium. Furthermore, the equilibrium distribution of country shares, the Lorenz curve, is *unique* (at least with a sufficiently large number of countries), and analytically tractable in the limit, as the number of countries grows unbounded.[3] Using this limit as an approximation allows us to study, among other things, what determines the shape of distribution and how various forms of globalization or technical change affect inequality across countries, and to evaluate the welfare effects of trade (e.g., when trade is Pareto-improving, and when it is not, what fractions of countries might lose from trade).

Section 2.1 introduces the baseline model, which assumes that all consumption goods are tradeable and all primary factors are in fixed supply. Then, section 2.2 derives a single-country

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[3]Whether the Lorenz curve is unique with any finite number of countries is an open question, although I conjecture that it is. The assumption that the set of tradeable goods vastly outnumbers the set of countries is not only realistic, but also crucial for ensuring a strict ranking across countries and the uniqueness of the Lorenz curve, in any stable equilibrium. If there were more countries than goods, as assumed in Matsuyama (1996), a closest precedent to the present paper, or if the number of countries and goods are in the same order of magnitude, some (though not all) countries could remain ex-post identical, forming a cluster of countries. Furthermore, the equilibrium distribution could fail to be unique. It is assumed that the set of tradeable goods is a continuum not only to simplify the analysis, but also to ensure that it outnumbers any (finite) set of countries.

(or autarky) equilibrium. Section 2.3 derives a stable equilibrium with any finite number of countries, whose associated Lorenz curve is characterized by the second-order difference equation with two terminal conditions. Section 2.4 explains why any equilibrium in which some countries remain identical ex-post is unstable. Section 2.5 shows that, as the number of countries grows unbounded, the Lorenz curve converges to the solution of the second-order differential equation with two terminal conditions, which is unique and analytically solvable. Armed with the explicit formula for the limit Lorenz curve, this subsection shows when the distribution is bimodal or satisfies a power-law. It also demonstrates that making local services more differentiated causes a Lorenz-dominant shift of the distribution, leading to a greater inequality across countries. Section 2.6 studies the welfare effects of trade, which turns out to depend on the heterogeneity of tradeable goods, measured by the Theil index of their dependence on local services. Section 3 discusses two extensions. In section 3.1, a fraction of the consumption goods are assumed to be nontradeable. This extension allows us to study the effects of globalization through trade in goods. In section 3.2, one of the primary factors is allowed to vary in supply either through factor mobility and accumulation. This extension not only generates the correlation between the capital-labor ratio and per capita income and TFP, but also it allows us to study the effects of technical change that increases the relative importance of human capital in production and of globalization through factor mobility.

2. Baseline Model:

2.1 Key Elements of the Model

The world consists of J (ex-ante) identical countries, where J is a positive integer. There may be multiple nontradeable primary factors of production, such as capital (K), labor (L), etc., which can be aggregated to a single composite as $V = F(K, L, …)$. For now, it is assumed that these component factors are in fixed supply and that the representative consumer of each country is endowed with the same quantity of V.

The representative consumer has Cobb-Douglas preferences over a continuum of tradeable consumption goods, indexed by $s \in [0,1]$. This can be expressed by an expenditure function, $E = \exp\left[\int_0^1 \log(P(s)) dB(s)\right] U$, where U is utility; $P(s) > 0$ the price of good-s; $B(s) =$

$\int_0^s \beta(u)du$ the expenditure share of goods in [0,*s*], satisfying $B'(s) = \beta(s) > 0$, $B(0) = 0$, and

$B(1) = 1$. By denoting the aggregate income by *Y*, the budget constraint is then written as

(1) $Y = \exp\left[\int_0^1 \log(P(s))dB(s)\right]U$.

Each tradeable good is produced competitively with CRS technology, using two types of nontradeable inputs. They are the (composite) primary factor of production and a composite of differentiated local producer services, aggregated by a symmetric CES, as in Dixit and Stiglitz (1977). The primary factor and the composite of local producer services are combined with a Cobb-Douglas technology with $\gamma(s) \in [0,1]$ being the share of local producer services in sector-*s*. The unit cost of producing each tradeable good can thus be expressed as

(2) $C(s) = \zeta(s)(\omega)^{1-\gamma(s)}\left[\int_0^n (p(z))^{1-\sigma} dz\right]^{\frac{\gamma(s)}{1-\sigma}} = \zeta(s)(\omega)^{1-\gamma(s)}\left[\int_0^n (p(z))^{-\frac{1}{\theta}} dz\right]^{-\theta\gamma(s)}$,

where ω is the price of the (composite) primary factor; *n* the range of differentiated producer services available in equilibrium; $p(z)$ the price of a variety $z \in [0,n]$; and $\zeta(s) > 0$ a scale parameter. The parameter, $\sigma > 1$, is the direct partial elasticity of substitution between each pair of services. It is notationally convenient to define $\theta = 1/(\sigma - 1) > 0$, which I shall call *the degree of differentiation*. What is crucial here is that the tradeable goods differ in their dependence on local services, $\gamma(s)$. With no loss of generality, we may order the tradeable goods so that $\gamma(s)$ is increasing in $s \in [0,1]$. For technical reasons, it is assumed to be continuously differentiable.

Monopolistic competition prevails in the local services sector. Each variety is supplied by a single firm, which uses $T(q) = f + mq$ units of the primary factor to supply *q* units so that the total cost is $\omega(f + mq)$, of which the fixed cost is ωf and ωm represents the marginal cost. As is well-known, each monopolistically competitive firm would set its price equal to $p(z) = (1+\theta)\omega m$ if unconstrained in this environment. This would mean that it might not be clear whether the effects of shifting $\theta = 1/(\sigma - 1) > 0$ should be attributed to a change in the degree of differentiation or a change in the mark-up rate. To separate these two conceptually, let us depart from the Dixit-Stiglitz specification by introducing a competitive fringe. That is, once a firm pays the fixed cost of supplying a particular variety, any other firms in the same country could supply its perfect substitute with the marginal cost equal to $(1+v)\omega m > \omega m$ without paying any

fixed cost, where $v > 0$ is the productivity disadvantage of the competitive fringe. When $v \leq \theta$, such competitive fringe forces the monopolistically competitive firm to charge a limit price,

(3) $p(z) = (1+v)\omega m,$ for all $z \in [0,n]$, where $0 < v \leq \theta$.

Note that this pricing rule generalizes the Dixit-Stiglitz formulation, as the latter is captured by the special case, $v = \theta$. This generalization is introduced merely to demonstrate that the main results are independent of v, when $v < \theta$, so that the effects of θ should be interpreted as those of changing the degree of differentiation, not the mark-up rate.[4]

From (3), the unit production cost of each tradeable good, (2), is simplified to:

(4) $C(s) = \zeta(s)\{(1+v)m\}^{\gamma(s)}(n)^{-\theta\gamma(s)}\omega$.

Eq. (4) shows that, given ω, a higher n reduces the unit production cost in all tradeables, which is nothing but productivity gains from variety, *a la* Ethier (1982a)-Romer (1987); that this effect is stronger for a larger θ; and that higher-indexed sectors gain more from this effect.

Since all the services are priced equally and enter symmetrically into production, $q(z) = q$ for all $z \in [0,n]$, and the profit of all service providers is given by $\pi(z) = pq - \omega(mq + f) = \omega(vmq - f)$ for all $z \in [0,n]$. Thus, they earn zero profit if and only if:

(5) $vmq = f,$

which holds in equilibrium due to the free entry-and-exit to the service sector.

Before proceeding, note that we may set,

(6) $B(s) = s$ for all $s \in [0,1]$,

so that $\beta(s) = 1$ for all $s \in [0,1]$ by choosing the tradeable goods indices, without further loss of generality.[5] In words, we measure the size of (a set of) sectors by the expenditure share of the goods produced in these sectors. With this indexing, the size of sectors whose γ is less than or equal to $\gamma(s)$ is equal to s, and a country's share in the world income is equal to the measure of the tradeables for which the country ends up having comparative advantage in equilibrium.

2.2   Single-Country (or Autarky) Equilibrium ($J = 1$)

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[4] This generalization of the Dixit-Stiglitz model to separate the roles of mark-ups and product differentiation has been used previously by, e.g. Matsuyama and Takahashi (1998) and Acemoglu (2009, Ch.12.4.4).

[5] To see this, starting from any indexing of the goods $s' \in [0,1]$ satisfying i) $\tilde{\gamma}(s') \in [0,1]$ is increasing in $s' \in [0,1]$,

ii) $\tilde{\beta}(s') > 0$ for $s' \in [0,1]$, and iii) $\int_0^1 \tilde{\beta}(s')ds' = 1$, re-index the goods by $s = \tilde{B}(s') \equiv \int_0^{s'} \tilde{\beta}(u)du$. Then,

$\gamma(s) \equiv \tilde{\gamma}(\tilde{B}^{-1}(s))$ is increasing in $s \in [0,1]$ and $ds = \tilde{\beta}(s')ds'$, so that $\beta(s) = 1$ for $s \in [0,1]$.

First, let us look at the equilibrium allocation of a one-country world ($J = 1$). This can also be viewed as the equilibrium allocation of a country in autarky in a world with multiple countries, which would serve as the benchmark for evaluating the welfare effects of trade.

Because of Cobb-Douglas preferences, the economy must produce all the consumption goods in the absence of trade, which means that their prices must be equal to their costs; that is,

(7) $P(s) = C(s) = \zeta(s)\{(1+v)m\}^{\gamma(s)}(n)^{-\theta\gamma(s)}\omega$ for all $s \in [0,1]$

Since the representative consumer spends $\beta(s)Y = Y$ on good-s, and sector-s spends $100\gamma(s)\%$ of its revenue on producer services, the total revenue of the producer services sector is

(8) $npq = n(1+v)m\omega q = \int_0^1 \gamma(s)\beta(s)Yds = \Gamma^A Y,$ where $\Gamma^A \equiv \int_0^1 \gamma(s)ds.$

Thus, in autarky, the share of the producer services in the aggregate income is equal to the average share of the producer services across all the consumption goods sectors.

Likewise, sector-s spends $100(1-\gamma(s))\%$ of its revenue on the primary factor. In addition, each service provider spends $\omega(f+mq)$ on the primary factor. Therefore, the total income earned by the (composite) primary factor is equal to:

(9) $\omega V = \int_0^1 (1-\gamma(s))\beta(s)Yds + n\omega(f+mq) = (1-\Gamma^A)Y + n\omega(f+mq).$

Combining (8) and (9) with the zero-profit condition (5) yields the equilibrium variety of services (and the number of service providers) as well as the aggregate income:

(10) $n^A = \Gamma^A \left(\dfrac{v}{1+v}\right)\left(\dfrac{V}{f}\right)$

(11) $Y^A = \omega^A V = \omega^A F(K,L,...).$

Eq. (10) shows that the equilibrium variety, n^A, is proportional to the share of producer services in the total expenditure, which is equal to Γ^A in autarky. Eq.(11) shows that, because free entry ensures zero profit, the aggregate income is accrued entirely to the primary factors. With all the primary factors, capital (K), labor (L), etc. being aggregated into the composite, $V = F(K,L,...)$, the equilibrium price of the composite factor, ω^A, is nothing but TFP, as measured in GDP accounting.

2.3   Stable Multi-Country Equilibrium ($J \geq 2$)

Let us now turn to the case with $J \geq 2$. Since these countries are ex-ante identical, they share the same values for all the exogenous parameters. However, the stability of equilibrium requires that no two countries share the same value of n, as explained later. This allows us to rank the countries such that $\{n_j\}_{j=1}^{J}$ is (strictly) increasing in j. (The subscript here indicates the rank, not the identity, of a country in a particular equilibrium.) Then, from (4), the relative cost between the j-th and the $(j+1)$-th countries,

$$\frac{C_j(s)}{C_{j+1}(s)} = \left(\frac{n_j}{n_{j+1}}\right)^{-\theta\gamma(s)}\left(\frac{\omega_j}{\omega_{j+1}}\right),$$

is increasing in s for any $j = 1, 2, ..., J-1$, for any combination of the factor prices $\{\omega_j\}_{j=1}^{J}$, as illustrated by upward-sloping curves in Figure 1. In words, a country with a more developed local support industry has comparative advantage in higher-indexed sectors, which rely more heavily on local services. Furthermore, $\{\omega_j\}_{j=1}^{J}$ must adjust in equilibrium so that each country becomes the lowest cost producer (and hence the exporter) of a positive measure of the tradeable goods.[6] This implies that a sequence, $\{S_j\}_{j=0}^{J}$, defined by $S_0 = 0$, $S_J = 1$, and

$$\frac{C_j(S_j)}{C_{j+1}(S_j)} = \left(\frac{n_j}{n_{j+1}}\right)^{-\theta\gamma(S_j)}\left(\frac{\omega_j}{\omega_{j+1}}\right) = 1 \qquad\qquad (j = 1, 2, ..., J-1),$$

is increasing in j.[7] As showed in Figure 1, the tradeable goods, [0,1], are partitioned into J subintervals of positive measure, (S_{j-1}, S_j), such that the j-th country becomes the lowest cost producer (and hence its sole producer and exporter) of $s \in (S_{j-1}, S_j)$.[8] Note also that the definition of $\{S_j\}_{j=1}^{J-1}$ can be rewritten to obtain:

(12) $$\frac{\omega_{j+1}}{\omega_j} = \left(\frac{n_{j+1}}{n_j}\right)^{\theta\gamma(S_j)} > 1. \qquad\qquad (j = 1, 2, ..., J-1)$$

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[6] Otherwise, the factor price would be zero for a positive fraction (at least $1/J$) of the world population.
[7] To see why, $S_j \geq S_{j+1}$ would imply $C_j(s) > \min\{C_{j-1}(s), C_{j+1}(s)\}$ for all $s \in [0,1]$, hence that the j-th country is not the lowest cost producers of any tradeable good, a contradiction.
[8] In addition, S_0 is produced and exported by the 1st country and S_J by the J-th country. The borderline sector, S_j ($j = 1, 2,…, J-1$), could be produced and exported by either j-th or $(j+1)$-th country or both. This type of indeterminacy is inconsequential and ignored in the following discussion.

so that $\left\{\omega_j\right\}_{j=1}^{J}$ is also increasing in j; that is, a country with a more developed local support industry (i.e., a higher j) is more productive due to the variety effect.

Since the j-th country specializes in (S_{j-1}, S_j), $100(S_j-S_{j-1})\%$ of the world income, Y^W, is spent on its tradeable sectors, and its sector-s in (S_{j-1}, S_j) spends $100\gamma(s)\%$ of its revenue on its local services. Thus, the total revenues of its local producer services sector is equal to

$$(13) \quad n_j p_j q_j = n_j(1+v)m\omega_j q_j = \left[\int_{S_{j-1}}^{S_j} \gamma(s)ds\right] Y^W = (S_j-S_{j-1})\Gamma_j Y^W, \qquad (j = 1, 2, ...,J)$$

where

$$(14) \quad \Gamma_j \equiv \Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds. \qquad (j = 1, 2, ...,J)$$

is the average of γ over (S_{j-1}, S_j). Note that $\left\{\Gamma_j\right\}_{j=1}^{J}$ is increasing in j, since γ is increasing in s.

Likewise, in the j-th country, sector-$s \in (S_{j-1}, S_j)$ spends $100(1-\gamma(s))\%$ of its revenue on its primary factor, and each service provider spends $\omega_j(f+mq_j)$ on its primary factor. Thus, the total income earned by the primary factor in the j-th country is equal to:

$$(15) \quad \omega_j V = (1- \Gamma_j)(S_j-S_{j-1})Y^W + n_j\omega_j(mq_j + f) \qquad (j = 1, 2, ...,J)$$

Combining (13) and (15) with the zero-profit condition (5) yields:

$$(16) \quad n_j = \Gamma_j\left(\frac{v}{1+v}\right)\left(\frac{V}{f}\right); \qquad (j = 1, 2, ...,J)$$

and

$$(17) \quad Y_j = \omega_j V = (S_j - S_{j-1})Y^W. \qquad (j = 1, 2, ...,J)$$

Because $\left\{\Gamma_j\right\}_{j=1}^{J}$ is increasing in j, eq.(16) shows that $\left\{n_j\right\}_{j=1}^{J}$ is also increasing in j, as has been assumed. Eq.(17) shows that ω_j represents TFP of the j-th poorest country, and $s_j \equiv S_j - S_{j-1}$, the size of the tradeable sectors in which this country has comparative advantage, is also equal to its share in the world income. Thus, $S_j = \sum_{k=1}^{j} s_k$ is the cumulative share of the j poorest countries in world income. By combining (12), (14), (16), and (17), we obtain:

Proposition 1: Let S_j be the cumulative share of the j poorest countries in world income.

Then, $\left\{S_j\right\}_{j=0}^{J}$ solves the second-order difference equation with two terminal conditions:

$$(18) \quad \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)} \right)^{\theta\gamma(S_j)} > 1 \text{ with } S_0 = 0 \text{ \& } S_J = 1, \text{ where } \Gamma(S_{j-1}, S_j) \equiv \frac{\int_{S_{j-1}}^{S_j} \gamma(s)ds}{S_j - S_{j-1}}.$$

Figure 2 illustrates a solution to eq. (18) by means of the Lorenz curve, $\Phi^J : [0,1] \to [0,1]$, defined by the piece-wise linear function, satisfying $\Phi^J(j/J) = S_j$. From this Lorenz curve, we can easily recover $\{s_j\}_{j=0}^J$, the distribution of the country shares in the world income and *vice versa*.[9] A few points deserve emphasis. First, because $\Gamma_j \equiv \Gamma(S_{j-1}, S_j)$ is increasing in *j*, $s_{j+1}/s_j \equiv (S_{j+1} - S_j)/(S_j - S_{j-1}) > 1$. Hence, the Lorenz curve is kinked at j/J for each $j = 1, 2, ..., J-1$. In other words, the ranking of the countries is strict.[10] Second, since both income and TFP are proportional to $s_j \equiv S_j - S_{j-1}$, the Lorenz curve here also represents the Lorenz curve for income and TFP. Third, we could also obtain the ranking of countries in other variables of interest that are functions of $\{s_j\}_{j=0}^J$. For example, the *j*-th country's share in world trade is equal to $\{s_j - (s_j)^2\}/\sum_{k=1}^J \{s_k - (s_k)^2\}$, which is increasing in *j*. The *j*-th country's trade dependence, defined by the volume of trade divided by its GDP, is equal to $1 - s_j$, which is decreasing in *j*.

2.4 Symmetry-Breaking: Instability of Equilibria without Strict Ranking of Countries

In characterizing the above equilibrium, we have started by imposing the condition that no two countries share the same value of *n*, and hence the countries could be ranked strictly so that $\{n_j\}_{j=1}^J$ is increasing in *j*, and then verified later that this condition holds in equilibrium. Indeed, there are also equilibria, in which some countries share the same value of *n*, and without strict ranking, $\{n_j\}_{j=1}^J$ is merely nondecreasing in *j*. For example, consider the case of $J = 2$ and suppose $n_1 = n_2$ in equilibrium. Then, from (4), $C_1(s)/C_2(s) = \omega_1/\omega_2$, which is independent of *s*. Thus, the condition under which each country produces a positive measure of goods is satisfied only if $C_1(s)/C_2(s) = \omega_1/\omega_2 = 1$. This means that, in this equilibrium, the consumers

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[9] This merely states that there is a one-to-one correspondence between the distribution of income shares and the Lorenz curve. With *J* ex-ante identical countries, there are *J*! (factorial) equilibria for each Lorenz curve.

[10] This is in sharp contrast to the model of Matsuyama (1996), which generates a non-degenerate distribution of income across countries, but with a clustering of countries that share the same level of income. The crucial difference is that the countries outnumber the tradeable goods in the model of Matsuyama (1996), while the tradeable goods outnumber the countries in the present model.

everywhere is indifferent as to which country they purchase tradeable goods from. In other words, the patterns of trade are indeterminate. *If* exactly 50% of the world income is spent on each country's tradeable goods sectors, and *if* this spending is distributed across the two countries in such a way that the local services sector of each country ends up receiving exactly $\Gamma^A/2$ fraction of the world spending, then free entry to this sector in each country would lead to $n_1 = n_2 = n^A$. Thus, the two countries remain identical ex-post. However, it is easy to see that this symmetric equilibrium, which replicates the autarky equilibrium in each country, is "sitting on the knife-edge", in that the required spending patterns described above must be met exactly in spite of the consumers' indifference. Furthermore, this equilibrium is unstable in that a small perturbation that causes $n_1 \neq n_2$ would lead to a change in the spending patterns that makes the profit of local service providers in the country with a higher (lower) *n* rise (fall), which makes *n* even higher (lower) in that country, pushing the world economy further away from the symmetric equilibrium. The logic of symmetry-breaking also carries over to the case of $J > 2$ with $n_j = n_{j+1}$ for some *j*, because that would imply $C_j(s)/C_{j+1}(s) = \omega_j/\omega_{j+1} = 1$ so that, for a positive measure of goods, the consumers would be indifferent between buying from the *j*-th or (*j*+1)-th country, which would generate the same knife-edge property. For this reason, we restrict ourselves only to the equilibrium with a strict ranking of countries.[11]

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[11] The logic behind the instability of equilibrium without strict ranking of countries is similar to that of the mixed strategy equilibrium in games of strategic complementarity, particularly the game of the battle of the sexes. The assumption of a finite number of countries is crucial, but the assumption of zero trade cost in tradeable goods is not. To understand the latter, consider the case of $J = 2$. Imagine that the two countries are ex-ante identical, but *repeatedly* hit by small random shocks such that the realized parameter values cause the ratio of n^A of the two countries to fluctuate over a small support around one, $[e^{-\varepsilon}, e^{\varepsilon}]$ with $\varepsilon > 0$. (For example, the relative size of the two countries, V_1/V_2, might fluctuate around one, due to small shocks to exogenous components of TFP.) Now, assume an iceberg trade cost, such that one unit of the good shipped shrinks to $e^{-\delta} < 1$ when it arrives, where $\delta > 0$. Then, one could show that, even with a small $\varepsilon > 0$, symmetry breaking occurs eventually if $\delta < \varepsilon\theta(\gamma(1)-\gamma(0))/2$. (The logic here should be familiar to those who are exposed to the notion of stochastic stability of dynamical systems with random perturbations, where the long run stability of equilibrium depends on the size of its basin of attraction.) This extension also suggests that the world undergo a symmetry-breaking bifurcation, when the trade cost declines from $\delta > \varepsilon\theta(\gamma(1)-\gamma(0))/2$ to $\delta < \varepsilon\theta(\gamma(1)-\gamma(0))/2$. Of course, this means that, with a small positive cost $\delta > 0$, infinitesimal perturbations $\varepsilon \to 0$ cannot break symmetry. However, this is a mere technicality with no substantive issue at stake. Symmetry-breaking captures the idea that the symmetric outcome is more vulnerable to small shocks than the asymmetric outcomes, so that the asymmetric outcomes are likely to be observed. What matters is that, the smaller the trade cost, smaller shocks are enough to break symmetry.

Incidentally, there are a couple of ways to extend the model that could ensure that equilibrium without strict ranking is unstable even to infinitesimal shocks in spite of a small trade cost. For example, the present model assumes for simplicity, like any standard Ricardian model, that the goods produced by different countries are perfect substitutes within each sector. This means that introducing a small iceberg trade cost causes a discontinuous shift in the demand across countries, which is why small (but not infinitesimally) shocks are needed for breaking symmetry. Instead, consider the Armington specification that the goods produced by different countries within each sector are

The symmetry-breaking mechanism that renders equilibrium without strict ranking unstable and leads to the emergence of strict ranking across ex-ante identical countries is a *two-way causality* between the patterns of trade and cross-country productivity differences. A country with a more developed local services sector not only has higher TFP, but also comparative advantage in tradeable sectors that depend more on local services. Having comparative advantage in such sectors means a larger local demand for such services, which leads to a more developed local services sector and hence higher TFP. Since tradeable goods differ in their dependence of local services, some countries end up becoming less productive and poorer than others. Although many similar symmetry-breaking mechanisms exist in the trade and geography literature, they are usually demonstrated in models with two countries or regions. One advantage of the present model is that, with a continuum of goods, the logic extends to any finite number of countries or regions.

2.5 **Equilibrium Lorenz Curve: Limit Case ($J \rightarrow \infty$)**

Even though eq. (18) fully characterizes the equilibrium distribution of country shares, it is not analytically solvable. Of course, one could try to solve it numerically. However, numerical methods are not useful for answering the question of the uniqueness of the solution or for determining how the solution depends on the parameters of the model. Instead, in spirit similar to the central limit theorem, let us approximate the equilibrium Lorenz curve with a large but a finite number of countries by $\lim_{J \to \infty} \Phi^J = \Phi$.[12] It turns out that, as $J \rightarrow \infty$, eq.(18) converges to the second-order differential equation with two terminal conditions, whose solution is unique and can be solved analytically. This allows us to study not only what determines the shape of the Lorenz curve, but also conduct various comparative statics, and to evaluate the welfare effects of trade.

Here's how to obtain the limit Lorenz curve, $\lim_{J \to \infty} \Phi^J = \Phi$. The basic strategy is to take Taylor expansions on both sides of eq. (18).[13] First, by setting $x = j/J$ and $\Delta x = 1/J$,

$$S_{j+1} - S_j = \Phi(x + \Delta x) - \Phi(x) = \Phi'(x)\Delta x + \Phi''(x)\frac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

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highly but not perfectly substitutable. This would make the property of the model continuous at zero trade cost, so that even infinitesimal perturbations would be enough for symmetry-breaking even in spite of a small trade cost.

[12] Note that this is different from assuming a continuum of countries, as eq.(18) is derived under the assumption that there are a finite number of countries.

[13] Initially, I obtained the limit by a different method, which involves repeated use of the mean value theorem. I am grateful to Hiroshi Matano for showing me this (more efficient) method.

$$S_j - S_{j-1} = \Phi(x) - \Phi(x - \Delta x) = \Phi'(x)\Delta x - \Phi''(x)\frac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

from which the LHS of eq. (18) can be written as:

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o\left(|\Delta x|\right).$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s)ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right)$$

$$\Gamma(S_j, S_{j-1}) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s)ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right)$$

from which the RHS of eq.(18) can be written as:

$$\left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)}\right)^{\theta\gamma(S_j)} = \left(1 + \frac{\gamma'(\Phi(x))}{\gamma(\Phi(x))}\Phi'(x)\Delta x + o\left(|\Delta x|\right)\right)^{\theta\gamma(\Phi(x))} = 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right).$$

By combining these, eq.(18) becomes:

$$1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o\left(|\Delta x|\right) = 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o\left(|\Delta x|\right).$$

By letting $\Delta x = 1/J \to 0$, this becomes:

(19) $\quad \dfrac{\Phi''(x)}{\Phi'(x)} = \theta\gamma'(\Phi(x))\Phi'(x)$.

Integrating (19) twice yields $\int_0^{\Phi(x)} e^{-\theta\gamma(s)}ds = c_1 + e^{c_0}x = \left[\int_0^1 e^{-\theta\gamma(u)}du\right]x$, where two integral

constants, c_0 and c_1, are pinned down by the two terminal conditions, $\Phi(0) = 0$ and $\Phi(1) = 1$.

This can be further rewritten as follows:

Proposition 2: The limit equilibrium Lorenz curve, $\lim_{J\to\infty} \Phi^J = \Phi$, is given by:

(20) $\qquad x = H\big(\Phi(x)\big) \equiv \int_0^{\Phi(x)} h(s)ds$, where $h(s) \equiv \dfrac{e^{-\theta\gamma(s)}}{\int_0^1 e^{-\theta\gamma(u)}du}$.

Figure 3 illustrates eq. (20). As shown in the left panel, $h(s)$ is positive and decreasing in $s \in [0,$

1]. Thus, its integral, $x = H(s)$, is increasing and concave. Furthermore, $h(s)$ is normalized in

such a way that $H(0) = 0$ and $H(1) = 1$, as shown in the right panel. Hence, its inverse function, the Lorenz curve, $s = \Phi(x) = H^{-1}(x)$ is increasing, convex, with $\Phi(0) = 0$ and $\Phi(1) = 1$.

It is also worth noting that the Lorenz curve may be viewed as the one-to-one mapping between a set of countries (on the *x*-axis) and a set of the goods they produce (on the *s*-axis).

From $s = \Phi(x) = H^{-1}(x)$, one could calculate GDP of the country at $100x\%$ (with World GDP normalized to one) by $y = \Phi'(x)$ and its cumulative distribution function (cdf), by $x = \Psi(y) \equiv (\Phi')^{-1}(y)$. Table illustrates one such calculation for an algebraically tractable one-parameter family of γ functions, which turns out to generate power-law (e.g., truncated Pareto) distributions.[14] Example 1 and Example 2 may be viewed as the limit cases of Example 3, as $\lambda \to 0$ and $\lambda \to \theta$, respectively. Note that, as λ varies from $-\infty$ to $+\infty$, the "power" in the probability density function (pdf), $\lambda/\theta - 2$, changes from $-\infty$ to $+\infty$. As $\lambda \to -\infty$, a smaller fraction of the consumer expenditure goes to the sectors that use local services more intensively. This means that just a small fraction of countries specialize in such "desirable" tradeable goods. As a result, the pdf declines more sharply in the upper end.

Table: Power-Law Examples

	Example 1: $\gamma(s) = s$	Example 2: $\gamma(s) = \log\left[1 + (e^\theta - 1)s\right]^{\frac{1}{\theta}}$	Example 3: $\gamma(s) = \log\left[1 + (e^\lambda - 1)s\right]^{\frac{1}{\lambda}}$; $(\lambda \neq 0; \neq \theta)$
Inverse Lorenz Curve $x = H(s)$	$\dfrac{1 - e^{-\theta s}}{1 - e^{-\theta}}$	$\log\left[1 + (e^\theta - 1)s\right]^{\frac{1}{\theta}}$	$\dfrac{\left[1 + (e^\lambda - 1)s\right]^{1 - \frac{\theta}{\lambda}} - 1}{e^{\lambda - \theta} - 1}$
Lorenz Curve: $s = \Phi(x) = H^{-1}(x)$	$\log\left[1 - (1 - e^{-\theta})x\right]^{-\frac{1}{\theta}}$	$\dfrac{e^{\theta x} - 1}{e^\theta - 1}$	$\dfrac{\left[1 + (e^{\lambda - \theta} - 1)x\right]^{\frac{\lambda}{\lambda - \theta}} - 1}{e^\lambda - 1}$
Cdf: $x = \Psi(y)$ $= (\Phi')^{-1}(y)$	$\dfrac{1}{1 - e^{-\theta}} - \dfrac{1}{\theta y}$	$\dfrac{1}{\theta}\log\left(\dfrac{e^\theta - 1}{\theta}y\right)$	$\dfrac{\left(\dfrac{y}{y_{Min}}\right)^{\frac{\lambda}{\theta} - 1} - 1}{e^{\lambda - \theta} - 1} = 1 - \dfrac{1 - \left(\dfrac{y}{y_{Max}}\right)^{\frac{\lambda}{\theta} - 1}}{1 - e^{\theta - \lambda}}$
Pdf: $\psi(y) = \Psi'(y)$	$\dfrac{1}{\theta y^2}$	$\dfrac{1}{\theta y}$	$\left[\dfrac{(\lambda/\theta) - 1}{(y_{Max})^{(\lambda/\theta)-1} - (y_{Min})^{(\lambda/\theta)-1}}\right](y)^{\frac{\lambda}{\theta} - 2}$
Support $[y_{Min}, y_{Max}]$	$\dfrac{1 - e^{-\theta}}{\theta} \leq y \leq \dfrac{e^\theta - 1}{\theta}$	$\dfrac{\theta}{e^\theta - 1} \leq y \leq \dfrac{\theta e^\theta}{e^\theta - 1}$	$\left(\dfrac{\lambda}{e^\lambda - 1}\right)\left(\dfrac{e^{\lambda - \theta} - 1}{\lambda - \theta}\right) \leq y \leq \left(\dfrac{\lambda}{e^\lambda - 1}\right)\left(\dfrac{e^{\lambda - \theta} - 1}{\lambda - \theta}\right)e^\theta$

Eq.(20) defines the equilibrium mapping between $\Phi(x)$ and $\gamma(s)$. One could thus use it to investigate when a symmetry-breaking mechanism of this kind leads to, say, a bimodal

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[14] In addition to being algebraically tractable, the power-law examples have some empirical appeal when "countries" are interpreted as "cities" or "metropolitan areas": see, e.g., Gabaix and Ioannides (2004). I'm grateful to Fabrizio Perri, who suggested to me to construct power-law examples.

distribution, as the narrative in much of this literature, ("core-periphery" or "polarization") seems to suggest. When the (increasing and continuously differentiable function) γ can be approximated by a two-step function, the corresponding pdf becomes bimodal. Thus, the world becomes polarized into the rich core and the poor periphery, when the tradeable goods can be classified into two categories in such a way that they are roughly homogeneous within each category.[15] Generally, a symmetry-breaking mechanism of this kind leads to N "clusters" of countries if γ can be approximated by an N-step function.[16]

Another advantage of Eq.(20) is that one could easily see the effect of changing θ, as illustrated by the arrows in Figure 3. To see this, note first that $\hat{h}(s) \equiv e^{-\theta\gamma(s)}$, the numerator of $h(s)$, satisfies $\partial^2 \log(\hat{h}(s))/\partial\theta\partial s = -\gamma'(s) < 0$. In words, it is *log-submodular* in θ and s.[17] Thus, a higher θ shifts the graph of $\hat{h}(s) \equiv e^{-\theta\gamma(s)}$ down everywhere but proportionately more at a higher s. Since $h(s)$ is a rescaled version of $\hat{h}(s)$ to keep the area under the graph unchanged, the graph of $h(s)$ is rotated "clockwise" by a higher θ, as shown in the left panel. This "single-crossing" in $h(s)$ implies that a higher θ makes the Lorenz curve more "curved" and move further away from the diagonal line, as shown in the right panel. In other words, a higher θ causes a Lorenz-dominant shift of the Lorenz curve. Thus, any Lorenz-consistent inequality measure, such as the generalized Kuznets Ratio, the Gini index, the coefficients of variations, etc. all agree that a higher θ leads to greater inequality.[18]

2.6 **Welfare Effects of Trade**

The mere fact that trade creates ranking of countries, making some countries poorer than others, does not necessarily imply that trade make them poorer. We need to compare the utility

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[15]Formally, consider a sequence of (increasing and continuously differentiable) γ functions that converges point-wise to a two-step function, $\gamma(s) = \gamma_L$ for $s \in [0, \bar{s}]$ and $\gamma(s) = \gamma_H > \gamma_L$ for $s \in (\bar{s}, 1]$. Then, the sequence of the corresponding cdf's converges to the cdf, $\Psi(y) = 0$ for $y < 1 - (1-\bar{s})(1 - e^{-\theta\Delta\gamma})$; $\Psi(y) = [1 + (1/\bar{s} - 1)e^{-\theta\Delta\gamma}]^{-1}$ for $1 - (1-\bar{s})(1 - e^{-\theta\Delta\gamma}) < y < 1 + \bar{s}(e^{\theta\Delta\gamma} - 1)$, and $\Psi(y) = 1$ for $y > 1 + \bar{s}(e^{\theta\Delta\gamma} - 1)$, where $\Delta\gamma \equiv \gamma_H - \gamma_L > 0$.

[16] Note that this is different from assuming that γ is a N-step function, which is equivalent to assume that there are N (a finite number of) tradeable goods. Then, the equilibrium distribution would not be unique; see Matsuyama (1996) for $N = 2$. To obtain the uniqueness, it is essential that γ is increasing, which means that the set of the tradeable goods is a continuum, and hence outnumbers the set of the countries for a large but finite number, J.

[17]See Topkis (1998) for mathematics of super-(and sub-)modularity and Costinot (2009) for a recent application to international trade.

[18]Likewise, any shift in $\gamma(s)$ that rotates $h(s)$ clockwise leads to greater inequality.

levels under trade and under autarky. From eq.(1), the welfare under autarky is

$\log(U^A) = \log(\omega^A V) - \int_0^1 \log(P^A(s)) ds$. Likewise, the welfare of the country that ends up being

the j-th poorest can be written as $\log(U_j) = \log(\omega_j V) - \int_0^1 \log(P(s)) ds$, where the tradeable goods

prices satisfy $\dfrac{P(s)}{P^A(s)} = \left(\dfrac{\omega_k}{\omega^A}\right)\left(\dfrac{n_k}{n^A}\right)^{-\theta\gamma(s)} = \left(\dfrac{\omega_k}{\omega^A}\right)\left(\dfrac{\Gamma_k}{\Gamma^A}\right)^{-\theta\gamma(s)}$ for $s \in (S_{k-1}, S_k)$ for $k = 1, 2, \ldots, J.$

Combining these equations yields

$$\log\left(\frac{U_j}{U^A}\right) = \log\left(\frac{\omega_j}{\omega^A}\right) - \int_0^1 \log\left(\frac{P(s)}{P^A(s)}\right) ds = \log\left(\frac{\omega_j}{\omega^A}\right) - \sum_{k=1}^{J}\left[\int_{S_{k-1}}^{S_k} \log\left(\frac{\omega_k}{\omega^A}\right) ds - \int_{S_{k-1}}^{S_k} \theta\gamma(s)\log\left(\frac{\Gamma_k}{\Gamma^A}\right) ds\right],$$

which can be further rewritten as follows:

Proposition 3 (*J*-country case): The country that ends up being the *j*-th poorest under trade gains from trade if and only if:

$$(21) \qquad \log\left(\frac{U_j}{U^A}\right) = \log\left(\omega_j / \prod_{k=1}^{J}(\omega_k)^{(S_k - S_{k-1})}\right) + \theta\sum_{k=1}^{J}\Gamma_k \log\left(\frac{\Gamma_k}{\Gamma^A}\right)(S_k - S_{k-1}) > 0.$$

Eq. (21) in Proposition 3 offers a decomposition of the welfare effects of trade. The first term is the country's TFP relative to the world average, and it is increasing in j, negative at $j = 1$ and positive at $j = J$. The second term captures the usual gains from trade (i.e., after controlling for the income and TFP differences across countries) and it is always positive.[19] However, aside from rather obvious statements like "a country gains from trade if the second term dominates the first term," or "a country gains from trade if its income (and TFP) ends up being higher than the world average," Proposition 3 offers little insight without an explicit solution for eq. (18).

As $J \to \infty$, the task of evaluating the overall welfare effect becomes greatly simplified. By setting $x^* = j/J$ and $x = k/J$ in eq. (21) and noting that $\omega_j / \omega_k \to \Phi'(x^*)/\Phi'(x)$ and

$S_k - S_{k-1} \to \Phi'(x)dx$ as $J \to \infty$, eq.(21) converges to:

$$\log\left(\frac{U(x^*)}{U^A}\right) = \int_0^1 \log\left(\frac{\Phi'(x^*)}{\Phi'(x)}\right)\Phi'(x)dx + \theta\int_0^1 \gamma(\Phi(x))\log\left(\frac{\gamma(\Phi(x))}{\Gamma_A}\right)\Phi'(x)dx.$$

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[19]Proof: Consider the convex maximization problem: $\underset{\{c_k\}_{k=1}^{J}}{Max} \sum_{k=1}^{J}\Gamma_k \log(c_k)(S_k - S_{k-1})$ s.t. $\sum_{k=1}^{J} c_k(S_k - S_{k-1}) \leq 1$.

Though $c_k = 1$ satisfies the constraint, its optimum is reached at $c_k = \Gamma_k / \Gamma^A$, so that $\sum_{k=1}^{J}\Gamma_k \log(\Gamma_k / \Gamma^A)(S_k - S_{k-1})$

$> \sum_{k=1}^{J}\Gamma_k \log(1)(S_k - S_{k-1}) = 0.$

Since eq. (19) implies $\log(\Phi'(x)) - \theta\gamma(\Phi(x)) = c_0$, this can be rewritten as:

$$\frac{1}{\theta}\log\left(\frac{U(x^*)}{U^A}\right) = \int_0^1 (\gamma(\Phi(x^*)) - \gamma(\Phi(x)))\Phi'(x)dx + \int_0^1 \gamma(\Phi(x))\log\left(\frac{\gamma(\Phi(x))}{\Gamma^A}\right)\Phi'(x)dx$$

$$= \int_0^1 (\gamma(s^*) - \gamma(s))ds + \int_0^1 \gamma(s)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds = \gamma(s^*) - \Gamma^A + \int_0^1 \gamma(s)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds .$$

To summarize:

Proposition 4 (Limit case; $J \rightarrow \infty$): The country that ends up being at $100x^*$ percentile under trade gains from trade if and only if:

(22) $\qquad \frac{1}{\theta}\log\left(\frac{U(x^*)}{U^A}\right) = \gamma(s^*) - \Gamma^A + \int_0^1 \gamma(s)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds > 0,$

where $s^* = \Phi(x^*)$ or $x^* = \Phi^{-1}(s^*)$.

As in Proposition 3, Proposition 4 offers a decomposition of the welfare effects of trade. The first term, $\gamma(s^*) - \Gamma^A = \gamma(\Phi(x^*)) - \Gamma^A$, represents the size of local service sector, which affects TFP of the economy, relative to the world average and relative to the autarky. This term is increasing in $s^* = \Phi(x^*)$, negative at $x^* = 0$ and positive at $x^* = 1$. The second term captures the usual gains from trade (i.e., controlling for the productivity differences across countries) and it is always positive.[20] This implies that $\gamma(s^*) = \gamma(\Phi(x^*)) \geq \Gamma^A$ is a sufficient condition that a country gains from trade. In fact, Proposition 4 allows us to say a lot more about the overall welfare effects of trade, as shown in the following two Corollaries.

Corollary 1: All countries gain from trade if and only if

(23) $\qquad 1 - \frac{\gamma(0)}{\Gamma^A} \leq \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds .$

Note that the sufficient and necessary condition under which all countries gain from trade, (23), depends solely on $\gamma(\bullet)$. In particular, it is independent of θ, which only plays a role of magnifying the gains and losses from trade. LHS of eq.(23) shows how much the share of the service sector, and hence productivity, declines in the country that ends up being the poorest. RHS of eq. (23) is the Theil index (or entropy), which measures the dispersion, of γ. This corollary thus states that trade is Pareto-improving (i.e., even the country that ends up being the

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[20]The proof is analogous to the one given in the previous footnote, and hence omitted.

poorest would benefit from trade) when the tradeable goods are sufficiently diverse, as measured by the Theil index of γ, and hence the gains from specialization (by making countries ex-post heterogeneous) is sufficiently large.

Corollary 2: Suppose that (23) fails. Then, for $s_c > 0$, defined by

$$\gamma(s_c) \equiv \Gamma^A \left[1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A} \right) \log \left(\frac{\gamma(s)}{\Gamma^A} \right) ds \right],$$

a): All countries producing and exporting goods $s \in [0, s_c)$ lose from trade, while all countries producing and exporting goods $s \in (s_c, 1]$ gain from trade.

b): The fraction of the countries that lose from trade, $x_c = H(s_c; \theta)$, is increasing in θ, and satisfies $\lim_{\theta \to 0} x_c = s_c$ and $\lim_{\theta \to \infty} x_c = 1$.

Corollary 2 is illustrated in the right panel of Figure 3. All countries that end up specializing in $[0, s_c)$ lose from trade and they account for x_c fraction of the world. Note that s_c depends solely on $\gamma(\bullet)$ and is independent of θ. This means that, as θ goes up and the Lorenz curve shifts, s_c remains unchanged and x_c goes up. As varying θ from 0 to ∞ (i.e., σ from ∞ to 1), x_c increases from s_c to 1. Thus, when γ is such that *some* countries lose from trade, *virtually all* countries would lose from trade as the Dixit-Stiglitz composite approaches Cobb-Douglas.

3. Two Extensions

This section reports two extensions conducted in Matsuyama (2011, Section 3).

3.1 Nontradeable Consumption Goods: Globalization through Trade in Goods

The first extension allows a fraction of the consumption goods within each sector to be nontradeable. This extension is used to examine the effects of globalization through trade in goods. Suppose that each sector-*s* produces many varieties, a fraction τ of which is tradeable and a fraction 1−τ is nontradeable, and that they are aggregated by Cobb-Douglas preferences.[21] The

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[21] This specification assumes that the share of local differentiated producer services in sector-*s* is $\gamma(s)$ for both nontradeables and tradeables. This assumption is made because, when examining the effect of globalization by changing τ, we do not want the distribution of γ across all tradeable consumption goods to change. However, for some other purposes, it would be useful to consider the case where the distribution of γ among nontradeable consumption goods differs systematically from those among tradeable consumption goods. For example, Matsuyama (1996) allows for such possibility to generate a positive correlation between per capita income and the nontradeable consumption goods prices across countries, similar to the Balassa-Samuelson effect.

expenditure function is now obtained by replacing $\log(P(s))$ with $\tau \log(P_T(s)) + (1-\tau)\log(P_N(s))$ for each $s \in [0,1]$, where $P_T(s) = Min\{C_j(s))\}$ is the price of each tradeable good in sector-s, common across all countries, $P_N(s) = C_j(s)$ is the price of each nontradeable good in sector-s, which is equal to the unit of cost of production in each country. Following the steps similar in section 2.3, one could show:

Proposition 5 (*J*-country case): Let S_j be the cumulative share of the j poorest countries in world income. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\tau\Gamma(S_j, S_{j+1}) + (1-\tau)\Gamma^A}{\tau\Gamma(S_{j-1}, S_j) + (1-\tau)\Gamma^A} \right)^{\theta\gamma(S_j)} > 1 \text{ with } S_0 = 0,\ S_J = 1,\ \Gamma(S_{j-1}, S_j) \equiv \frac{\int_{S_{j-1}}^{S_j} \gamma(s)ds}{S_j - S_{j-1}}$$

Then, following the same steps in section 2.5, one could show:

Proposition 6 (Limit Case; $J \to \infty$): The limit equilibrium Lorenz curve in GDP and TFP, $\lim_{J \to \infty} \Phi^J = \Phi$, is given by

$$x = H\big(\Phi(x); g\big) \equiv \int_0^{\Phi(x)} h(s; g)ds,\ \text{where } h(s; g) \equiv \frac{\left[\left(1 + g\gamma(s)/\Gamma^A\right)^{\Gamma^A/g} e^{-\gamma(s)} \right]^{\theta}}{\int_0^1 \left[\left(1 + g\gamma(u)/\Gamma^A\right)^{\Gamma^A/g} e^{-\gamma(u)} \right]^{\theta} du};\ g \equiv \frac{\tau}{1-\tau} > 0.$$

Again, Figure 3 illustrates the solution. For each $g \equiv \tau/(1-\tau) > 0$, the left panel shows $h(s; g)$ and the right panel $H(s; g)$. It is also easy to verify $\lim_{\tau \to 0} h(s; g) = \lim_{g \to 0} h(s; g) = 1$. Thus, as $\tau \to 0$, each country converges to the same single-country (autarky) equilibrium and hence the Lorenz curve converges to the diagonal line, and inequality disappears. Likewise, $\lim_{g \to \infty} h(s; g) = h(s) \equiv$

$e^{-\theta\gamma(s)} / \left[\int_0^1 e^{-\theta\gamma(u)} du \right]$. Thus, as $\tau \to 1$, it converges to the one given in Proposition 2.

Indeed, a higher τ, as well as a higher θ, causes a Lorenz-dominant shift, as illustrated by the arrows in Figure 3. To see this, one just need to check that the numerator of $h(s; g)$,

$\hat{h}(s; g) \equiv \left[\left(1 + g\gamma(s)/\Gamma^A\right)^{\Gamma^A/g} e^{-\gamma(s)} \right]^{\theta}$, is *log-submodular* in g and s (and in θ and s). This means that both a higher τ and a higher θ make the graph of $h(s; g)$ rotate "clockwise," and hence the

Lorenz curve becomes more "curved" and moves away from the diagonal line. This result thus suggests that globalization through trade in goods leads to greater inequality across countries.

3.2 Variable Factor Supply: Effects of Factor Mobility and/or Factor Accumulation

The second extension allows variable supply in one of the components in the composite of primary factors. This extension not only generates the correlation between the capital-labor ratio and income and productivity, but also allows us to examine the effects of technical change that increases importance of human capital or of globalization through factor mobility.

Returning to the case where $\tau = 1$, let us now allow the available amount of the composite primary factors, V, to vary across countries by endogenizing the supply of one of the component factors, K, as follows:

(24) $V_j = F(K_j, L)$ with $\omega_j F_K(K_j, L) = \rho.$

where $F_K(K_j, L)$ is the first derivative of F with respect to K, satisfying $F_{KK} < 0$. In words, the supply of K in the j-th country responds to its TFP, ω_j, such that its factor price is equalized across countries at a common value, ρ. This can be justified in two different ways.

A. Factor Mobility: Imagine that L represents (a composite of) factors that are immobile across borders and K represents (a composite of) factors that are freely mobile across borders, which seek higher return until its return is equalized in equilibrium.[22] According to this interpretation, ρ is an equilibrium rate of return determined endogenously, although it is not necessary to solve for it in order to derive the Lorenz curve.[23]

B. Factor Accumulation: Reinterpret the structure of the economy as follows. Time is continuous. All the tradeable goods, $s \in [0,1]$, are intermediate inputs that goes into the production of a single final good, Y_t, with the Cobb-Douglas function, $Y_t = \exp\left[\int_0^1 \log(X_t(s))ds\right]$ so that its unit cost is $\exp\left[\int_0^1 \log(P_t(s))ds\right]$. The representative agent in each country consumes

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[22]Which factors should be viewed as mobile or not depends on the context. If "countries" are interpreted as smaller geographical units such as "metropolitan areas," K may include not only capital but also labor, with L representing "land." Although labor is commonly treated as immobile in the trade literature, we will later consider the effects of globalization via factor mobility, in which case certain types of labor should be included among mobile factors.

[23]Also, $Y_j = V_j = \omega_j F(K_j, L)$ should be now interpreted as GDP of the economy, not GNP, and K_j is the amount of K used in the j-th country, not the amount of K owned by the representative agent in the j-th country. This also means that the LHS of the budget constraint in the j-th country should be its GNP, not its GDP (Y_j). However, calculating the distributions of GDP (Y_j), TFP (ω_j), and K_j/L does not require to use the budget constraint for each country, given that all consumption goods are tradeable ($\tau = 1$). The analysis would be more involved if $\tau < 1$.

and invests the final good to accumulate K_t, so as to maximize $\int_0^\infty u(C_t)e^{-\rho t}dt$ s.t. $Y_t = C_t + \dot{K}_t$,

where ρ is the subjective discount rate common across countries. Then, the steady state rate of return on K is equalized at ρ.[24] According to this interpretation, K may include not only physical capital but also human capital, and the Lorenz curve derived below represents steady state inequality across countries.

With this modification and with $V = F(K, L) = AK^\alpha L^{1-\alpha}$, with $0 < \alpha < 1-1/\sigma = 1/(1+\theta)$, Matsuyama (2011, Section 3.2) shows:

Proposition 7 (*J*-country case): Let $\Phi^J : [0,1] \to [0,1]$ denote the Lorenz curve in *Y/L* and in

K/L, where $\Phi^J(j/J) = S_j$. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{S_{j+1}-S_j}{S_j-S_{j-1}} = \left(\frac{\Gamma(S_j,S_{j+1})}{\Gamma(S_{j-1},S_j)}\right)^{\frac{\theta\gamma(S_j)}{1-\alpha-\alpha\theta\gamma(S_j)}} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1, \text{ where } \Gamma(S_{j-1},S_j) \equiv \frac{\int_{S_{j-1}}^{S_j}\gamma(s)ds}{S_j - S_{j-1}}.$$

Note that Φ^J here represents the Lorenz curve in *Y/L* and in *K/L*, not in TFP. The distribution of TFP can be obtained from that of *Y/L* (or *K/L*), with $\omega_{j+1}/\omega_j = (Y_{j+1}/Y_j)^{1-\alpha}$. Following the steps similar to section 2.5,

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Proposition 8 (Limit Case; *J* → ∞): $\lim_{J\to\infty}\Phi^J = \Phi$, is given by:

$$x = H(\Phi(x);\alpha) \equiv \int_0^{\Phi(x)} h(s;\alpha)ds, \text{ where } h(s;\alpha) \equiv \frac{\left(1-\frac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}}{\int_0^1\left(1-\frac{\alpha\theta}{1-\alpha}\gamma(u)\right)^{1/\alpha}du}.$$

Again, Figure 3 illustrates the solution. For each $\alpha < 1-1/\sigma = 1/(1+\theta)$, the left panel shows

$h(s;\alpha)$ and the right panel $H(s;\alpha)$. Since $\lim_{\alpha\to 0}h(s;\alpha) = h(s) \equiv e^{-\theta\gamma(s)}/\left[\int_0^1 e^{-\theta\gamma(u)}du\right]$, the

solution converges to the one in Proposition 2, as $\alpha \to 0$.

Indeed, a higher α, as well as a higher θ, causes a Lorenz-dominant shift, as illustrated by the arrows in Figure 3. The reasoning should be familiar by now. The numerator of $h(s;\alpha)$,

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[24]The intertemporal resource constraint assumes not only that *K* is immobile but also that international lending and borrowing is not possible. Of course, these restrictions are not binding in steady state, because the rate of return is equalized across countries at ρ.

$\hat{h}(s;\alpha) \equiv \left(1 - [\alpha/(1-\alpha)]\theta\gamma(s)\right)^{1/\alpha}$, is *log-submodular* in α and s (and in θ and s). Thus, a higher α (and a higher θ) makes the graph of $h(s;\alpha)$ rotate "clockwise," and hence a Lorenz-dominant shift, as shown in the right panel. This result suggests that skill-biased technological change that increases the share of human capital and reduces the share of raw labor in production, or globalization through trade in some factors, both of which can be interpreted as an increase in α, could lead to greater inequality across countries.

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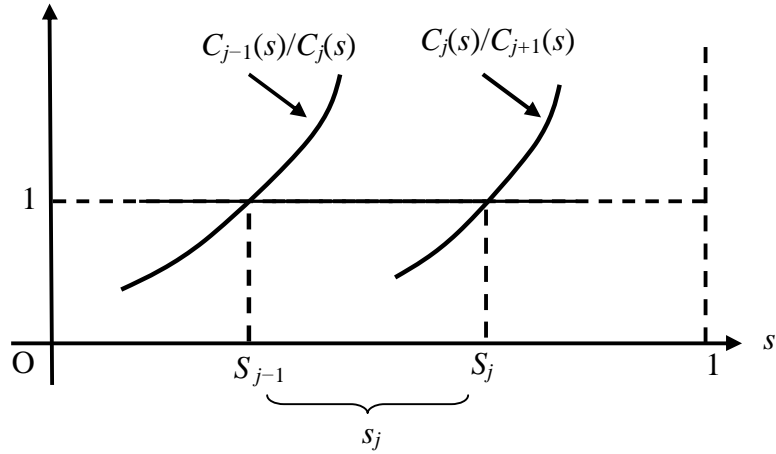Figure 1: Comparative Advantage and Patterns of Trade in the *J*-country World

Figure 2: Equilibrium Lorenz curve, Φ^J: A Graphic Illustration for $J = 4$
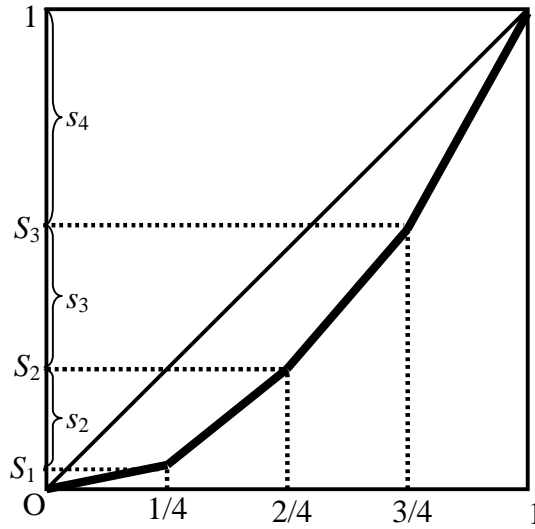

Figure 3: Limit Equilibrium Lorenz Curve, $\Phi(x)$, and its Lorenz-dominant Shift